Galactic dark matter as a bulk effect on the brane

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Abstract

The behavior of the angular velocity of a test particle moving in a stable circular orbit in the vacuum on the brane is considered. In the brane world scenario, the four dimensional effective Einstein equation acquire extra terms, called dark radiation and dark pressure, respectively, which arise from the embedding of the 3-brane in the bulk. A large number of independent observations have shown that the rotational velocities of test particles gravitating around galaxies tend, as a function of the distance from the galactic center, toward constant values. By assuming a constant tangential velocity, the general solution of the vacuum gravitational field equations on the brane can be obtained in an exact analytic form. This allows us to obtain the explicit form of the projections of the bulk Weyl tensor on the brane, and the equation of state of the dark pressure as a function of the dark radiation. The physical and geometrical quantities are expressed in terms of observable/measurable parameters, like the tangential velocity, the baryonic mass and the radius of the galaxy. We also analyze the dynamics of test particles by using methods from the qualitative analysis of dynamical systems, by assuming a simple linear equation of state for the dark pressure. The obtained results provide a theoretical framework for the observational testing at the extra-galactic scale of the predictions of the brane world models.

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I. INTRODUCTION

The problem of the dark matter is a long standing problem in modern astrophysics. Two important observational issues, the behavior of the galactic rotation curves and the mass discrepancy in clusters of galaxies led to the necessity of considering the existence of dark matter at a galactic and extra-galactic scale.

The rotation curves of spiral galaxies [1] are one of the best evidences showing the problems Newtonian gravity and/or standard general relativity have to face on the galactic/intergalactic scale. In these galaxies neutral hydrogen clouds are observed at large distances from the center, much beyond the extent of the luminous matter. Since these clouds move in circular orbits with velocity $v_{tg}(r)$, the orbits are maintained by the balance between the centrifugal acceleration v_{tg}^2/r and the gravitational attraction $GM(r)/r^2$ of the total mass $M(r)$ contained within the orbit. This allows the expression of the mass profile of the galaxy in the form $M(r) = rv_{tg}^2/G$.

Observations show that the rotational velocities increase near the center of the galaxy, in agreement with the theory, but then remain nearly constant at a value of $v_{tg\infty} \sim 200-300$ km/s [1], which leads to a mass profile $M(r) = rv_{tg\infty}^2/G$. Consequently, the mass within a distance r from the center of the galaxy increases linearly with r, even at large distances where very little luminous matter has been detected.

The second astrophysical evidence for dark matter comes from the study of the clusters of galaxies. The total mass of a cluster can be estimated in two ways. Knowing the motions of its member galaxies, the virial theorem gives one estimate, M_{VT}, say. The second is obtained by separately estimating the mass of each individual members, and summing these masses, to give a total baryonic mass M_B. Almost without exception it is found that M_{VT} is considerably greater than M_B, $M_{VT} > M_B$, typical values of M_{VT}/M_B being about 20-30 [1].

This behavior of the galactic rotation curves and of the virial mass of galaxy clusters is usually explained by postulating the existence of some dark (invisible) matter, distributed in a spherical halo around the galaxies. The dark matter is assumed to be a cold, pressureless medium. There are many possible candidates for dark matter, the most popular ones being the weakly interacting massive particles (WIMP) (for a review of the particle physics aspects of the dark matter see [2]). Their interaction cross sections with normal baryonic matter, while extremely small, are expected to be non-zero, and we may expect to detect

them directly.

It has also been suggested that the dark matter in the Universe might be composed of super-heavy particles, with mass $\geq 10^{10}$ GeV. But observational results show that the dark matter can be composed of super-heavy particles only if these interact weakly with normal matter, or if their mass is above 10^{15} GeV [3]. Scalar fields or other long range coherent fields coupled to gravity have also intensively been used to model galactic dark matter [4].

A general analysis of the possibility of an alternative four-dimensional gravity theory explaining the dynamics of galactic systems without dark matter was performed by Zhytnikov and Nester [5]. From very general assumptions about the structure of a relativistic gravity theory (the theory is metric, and invariant under general coordinates transformation, has a good linear approximation, it does not possess any unusual gauge freedom and it is not a higher derivative gravity) a general expression for the metric to order $(v/c)^2$ has been derived. This allows to compare the predictions of the theory with various experimental data: the Newtonian limit, light deflection and retardation, rotation of galaxies and gravitational lensing. The general conclusion of this study is that the possibility for any four-dimensional gravity theory to explain the behavior of galaxies without dark matter is rather improbable.

However, up to now no non-gravitational evidence for dark matter has been found, and no direct evidence or annihilation radiation from it has been observed yet.

Therefore, it seems that the possibility that Einstein's (and the Newtonian) theory of gravity breaks down at the scale of galaxies cannot be excluded *a priori*. Several theoretical models, based on a modification of Newton's law or of general relativity, have been proposed to explain the behavior of the galactic rotation curves [6].

The idea of embedding our Universe in a higher dimensional space has attracted a considerable interest recently, due to the proposal by Randall and Sundrum [7] that our four-dimensional (4d) spacetime is a three-brane, embedded in a 5d spacetime (the bulk). According to the brane world scenario, the physical fields (electromagnetic, Yang-Mills etc.) in our 4d Universe are confined to the three brane. Only gravity can freely propagate in both the brane and bulk spacetimes, with the gravitational self-couplings not significantly modified. Even if the fifth dimension is uncompactified, standard 4d gravity is reproduced on the brane. Hence this model allows the presence of large, or even infinite non-compact extra dimensions. Our brane is identified to a domain wall in a 5d anti-de Sitter spacetime. For a review of the dynamics and geometry of brane Universes, see e.g. [8].

Due to the correction terms coming from the extra dimensions, significant deviations from the standard Einstein theory occur in brane world models at very high energies [9]. Gravity is largely modified at the electro-weak scale of about 1 TeV. The cosmological and astrophysical implications of the brane world theories have been extensively investigated in the physical literature [10, 11].

The static vacuum gravitational field equations on the brane depend on the generally unknown Weyl stresses, which can be expressed in terms of two functions, called the dark radiation U and the dark pressure P terms (the projections of the Weyl curvature of the bulk, generating non-local brane stresses) [8, 12, 13].

Several classes of spherically symmetric solutions of the static gravitational field equations in the vacuum on the brane have been obtained in [16, 17, 18]. As a possible physical application of these solutions the behavior of the angular velocity v_{tg} of the test particles in stable circular orbits has been considered [17, 18]. The general form of the solution, together with two constants of integration, uniquely determines the rotational velocity of the particle. In the limit of large radial distances, and for a particular set of values of the integration constants the angular velocity tends to a constant value. This behavior is typical for massive particles (hydrogen clouds) outside galaxies [1], and is usually explained by postulating the existence of the dark matter.

Thus, the rotational galactic curves can be naturally explained in brane world models, without introducing any additional hypothesis. The galaxy is embedded in a modified, spherically symmetric geometry, generated by the non-zero contribution of the Weyl tensor from the bulk. The extra terms, which can be described in terms of the dark radiation term U and the dark pressure term P, act as a "matter" distribution outside the galaxy. The particles moving in this geometry feel the gravitational effects of U, which can be expressed in terms of an equivalent mass (the dark mass) M_U. The dark mass is linearly increasing with the distance, and proportional to the baryonic mass of the galaxy, $M_U(r) \approx M_B(r/r_0)$ [17].

The exact galactic metric, the dark radiation and the dark pressure in the flat rotation curves region in the brane world scenario has been obtained in [18].

Similar interpretations of the dark matter as bulk effects have been also considered in [19], where it was also shown that it is possible to model the X-ray profiles of clusters of galaxies, without the need for dark matter.

In the brane world scenario, the fundamental scale of gravity is not the Planck scale, but

another scale which may be at the TeV level. The gravitons propagating through the bulk space give rise to a Kaluza-Klein tower of massive gravitons on the brane. These gravitons couple to the energy-momentum term of the standard model fields, and could be produced under the appropriate circumstances as real or virtual particles.

Another important effect that is expected in the brane world models is the presence of brane fluctuations, since rigid objects do not exist in the relativistic theory. It was proposed that, in the context of brane world scenarios with low tension $\tau = f^4$, massive brane fluctuations are natural dark matter candidates, called branons [20]. The different possibilities for branons as dark matter candidates, the parameter region in which branons behave as collisionless thermal relics (WIMPs), either cold or hot (warm) together with less standard scenarios in which they are strongly self interacting or produced non-thermally, and the possibilities of obtaining branons in hadron accelerators have been considered in [21]. Another possibility to test these models is through the annihilation of branons into photon or electron-positron pairs in the galactic halos, which could possibly be detected by gamma-ray or anti-matter detectors.

A possible brane world model, which tries to give a justification for the hypothesis of the scalar field origin for the dark matter was proposed by [22]. The model contains two branes, on one of the branes lives the matter of the standard model of particles but on the other one, only spin-0 fundamental interactions are present. In the model the spin-0 fields are the dark matter. The scalar field contains an effective pressure, which avoids the collapse of the field fluctuations, implying that scalar field dominated objects, like galaxies, may contain a flat density profile in the center.

Gravitational lensing and the study of the light deflection by black holes and galaxies is an important physical effect that could provide specific signatures for testing the brane world models (for a review of the gravitational lensing by brane world black holes see [23]). Observables related to the relativistic images of strong field gravitational lensing could in principle be used to distinguish between different brane world black hole metrics in future observations. The strong field limit approach was used in [24] to investigate the gravitational lensing properties of brane world black holes, and the lensing observables for some candidate brane world black hole metrics have been compared with those for the standard Schwarzschild case.

Brane World black holes could have significantly different lensing observational signatures

as compared to the Schwarzschild black holes. The bending angle predicted by the brane world models is much larger than that predicted by standard general relativistic and dark matter models [23, 24]. The deflection of photons and the bending angle of light in the constant tangential velocity region on the brane was considered in [18, 19]. The bending angle predicted by the brane world models is much larger than that predicted by standard general relativistic and dark matter models. Therefore, the study of the gravitational lensing could discriminate between the different dynamical laws proposed to model the motion of particles at the galactic level and the standard dark matter models.

Generally, the vacuum field equations on the brane can be reduced to a system of two ordinary differential equations, which describe all the geometric properties of the vacuum as functions of the dark pressure and dark radiation terms [16]. In order to close the system of vacuum field equations on the brane a functional relation between these two quantities is necessary.

Hence, a first possible approach to the study of the vacuum brane consists in adopting an explicit equation of state for the dark pressure as a function of the dark radiation. A second method consist in closing the field equations on the brane by imposing the condition of the constancy of the rotational velocity curves for particle in stable orbits.

It is the purpose of the present paper to consider the general behavior of the vacuum gravitational field equations in the brane world model in the region of constant tangential rotational velocity of test particles in stable circular orbits. Physically, this situation is characteristic for particles gravitating in circular orbits around the galactic center [1]. As a first step in our study we derive, under the assumption of spherical symmetry, the basic equations describing the structure of the vacuum on the brane, and the equation giving the behavior of the tangential velocity of the test particles as functions of the dark radiation and of the dark pressure.

To obtain the tangential velocity of test particles we analyze the motion in an effective potential, also containing the effects of the bulk. By assuming that the brane is a fixed point of the bulk, we derive the equation giving the tangential velocity as a function of the 00 component of the metric tensor only. In the region of constant tangential velocities, the general solutions of the gravitational equations can be obtained in an exact analytic form.

This allows us to obtain the explicit form of the projections of the bulk Weyl tensor on the brane (the dark radiation and the dark pressure), as well as the components of the metric

tensor. The mass associated to the dark radiation (the dark mass) has a similar behavior as the dark matter at the galactic scale – it is a linearly increasing function of the distance, and it is proportional to the square of the tangential velocity. This result strongly supports the interpretation of the usual "dark matter" as a bulk effect. All the physical and geometrical quantities in our model are expressed in terms of observable/measurable parameters, like the tangential velocity, the baryonic mass and the radius of the galaxy. Some astrophysical applications of our model, including the possibilities of its observational testing, are also briefly considered.

Since generally the structure equations of the vacuum on the brane cannot be solved exactly, we shall analyze the dynamics of test particles by using methods from the qualitative analysis of dynamical systems. In particular, we will investigate the general behavior of the tangential velocity on the brane by assuming a simple linear equation of state for the dark pressure. In this case generally we cannot reproduce the observed behavior of the test particles.

The present paper is organized as follows. The field equations for the vacuum and the tangential velocity of a test particle on the brane are written down in Sections II and III. The general solution of the field equations in the constant tangential velocity region is obtained in Section IV. The qualitative analysis of the field equations for a linear equation of state for the dark pressure is performed in Section V. We discuss and conclude our results in Section VI.

II. THE FIELD EQUATIONS AND THE TANGENTIAL VELOCITY OF TEST PARTICLES FOR STATIC, SPHERICALLY SYMMETRIC VACUUM BRANES

A. The field equations in the brane world models

We start by considering a five dimensional (5d) spacetime (the bulk), with a single four-dimensional (4d) brane, on which matter is confined, only gravity can probe the extra dimensions. The 4d brane world $({}^{(4)}M, g_{\mu\nu})$ is located at a hypersurface $\left(B\left(X^{A}\right)=0\right)$ in the 5d bulk spacetime $({}^{(5)}M, g_{AB})$, of which coordinates are described by $X^{A}, A=0,1,\ldots,4$. The induced 4d coordinates on the brane are $x^{\mu}, \mu=0,1,2,3$.

The action of the system is given by [9]

$$S = S_{bulk} + S_{brane}, \tag{1}$$

where

$$S_{bulk} = \int_{^{(5)}M} \sqrt{-^{(5)}g}\left[\frac{1}{2k_5^2}{}^{(5)}R + {}^{(5)}L_m + \Lambda_5\right] d^5X, \tag{2}$$

and

$$S_{brane} = \int_{^{(4)}M} \sqrt{-^{(5)}g}\left[\frac{1}{k_5^2}K^{\pm} + L_{brane}\left(g_{\alpha\beta},\psi\right) + \lambda_b\right] d^4x, \tag{3}$$

where $k_5^2 = 8\pi G_5$ is the 5d gravitational constant, ${}^{(5)}R$ and ${}^{(5)}L_m$ are the 5d scalar curvature and the matter Lagrangian in the bulk, $L_{brane}\left(g_{\alpha\beta},\psi\right)$ is the 4d Lagrangian, which is given by a generic functional of the brane metric $g_{\alpha\beta}$ and of the matter fields ψ, $K^{\pm}$ is the trace of the extrinsic curvature on either side of the brane, and Λ_5 and λ_b (the constant brane tension) are the negative vacuum energy densities in the bulk and on the brane, respectively.

The Einstein field equations in the bulk are given by [9]

$${}^{(5)}G_{IJ} = k_5^2 {}^{(5)}T_{IJ}, \qquad {}^{(5)}T_{IJ} = -\Lambda_5 {}^{(5)}g_{IJ} + \delta(B)\left[-\lambda_b {}^{(5)}g_{IJ} + T_{IJ}\right], \tag{4}$$

where

$${}^{(5)}T_{IJ} \equiv -2\frac{\delta^{(5)}L_m}{\delta^{(5)}g^{IJ}} + {}^{(5)}g_{IJ}{}^{(5)}L_m, \tag{5}$$

is the energy-momentum tensor of bulk matter fields, while $T_{\mu\nu}$ is the energy-momentum tensor localized on the brane and which is defined by

$$T_{\mu\nu} \equiv -2\frac{\delta L_{brane}}{\delta g^{\mu\nu}} + g_{\mu\nu}\, L_{brane}. \tag{6}$$

The delta function $\delta\left(B\right)$ denotes the localization of brane contribution. In the 5d space-time a brane is a fixed point of the Z_2 symmetry. The basic equations on the brane are obtained by projections onto the brane world. The induced 4d metric is $g_{IJ} = {}^{(5)}g_{IJ} - n_I n_J$, where n_I is the space-like unit vector field normal to the brane hypersurface ${}^{(4)}M$. In the following we assume ${}^{(5)}L_m = 0$.

Assuming a metric of the form $ds^2 = (n_I n_J + g_{IJ})dx^I dx^J$, with $n_I dx^I = d\chi$ the unit normal to the $\chi =$ constant hypersurfaces and g_{IJ} the induced metric on $\chi =$ constant hypersurfaces, the effective 4d gravitational equation on the brane (the Gauss equation), takes the form [9]:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + k_4^2 T_{\mu\nu} + k_5^4 S_{\mu\nu} - E_{\mu\nu}, \tag{7}$$

where $S_{\mu\nu}$ is the local quadratic energy-momentum correction

$$S_{\mu\nu} = \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T_{\mu}{}^{\alpha} T_{\nu\alpha} + \frac{1}{24} g_{\mu\nu} \left(3 T^{\alpha\beta} T_{\alpha\beta} - T^2 \right), \tag{8}$$

and $E_{\mu\nu}$ is the non-local effect from the free bulk gravitational field, the transmitted projection of the bulk Weyl tensor C_{IAJB}, $E_{IJ} = C_{IAJB} n^A n^B$, with the property $E_{IJ} \to E_{\mu\nu} \delta_I^{\mu} \delta_J^{\nu}$ as $\chi \to 0$. We have also denoted $k_4^2 = 8\pi G$, with G the usual 4d gravitational constant.

The 4d cosmological constant, Λ, and the 4d coupling constant, k_4, are related by $\Lambda = k_5^2 (\Lambda_5 + k_5^2 \lambda_b^2 / 6)/2$ and $k_4^2 = k_5^4 \lambda_b / 6$, respectively. In the limit $\lambda_b^{-1} \to 0$ we recover standard general relativity [9].

The Einstein equation in the bulk and the Codazzi equation also imply the conservation of the energy-momentum tensor of the matter on the brane, $D_{\nu} T_{\mu}{}^{\nu} = 0$, where D_{ν} denotes the brane covariant derivative. Moreover, from the contracted Bianchi identities on the brane it follows that the projected Weyl tensor obeys the constraint $D_{\nu} E_{\mu}{}^{\nu} = k_5^4 D_{\nu} S_{\mu}{}^{\nu}$.

The symmetry properties of $E_{\mu\nu}$ imply that in general we can decompose it irreducibly with respect to a chosen 4-velocity field u^{μ} as [12]

$$E_{\mu\nu} = -k^4 \left[U \left(u_{\mu} u_{\nu} + \frac{1}{3} h_{\mu\nu} \right) + P_{\mu\nu} + 2 Q_{(\mu} u_{\nu)} \right], \tag{9}$$

where $k = k_5/k_4$, $h_{\mu\nu} = g_{\mu\nu} + u_{\mu} u_{\nu}$ projects orthogonal to u^{μ}, the "dark radiation" term $U = -k^{-4} E_{\mu\nu} u^{\mu} u^{\nu}$ is a scalar, $Q_{\mu} = k^{-4} h_{\mu}^{\alpha} E_{\alpha\beta} u^{\beta}$ is a spatial vector and $P_{\mu\nu} = -k^{-4} \left[h_{(\mu}{}^{\alpha} h_{\nu)}{}^{\beta} - \frac{1}{3} h_{\mu\nu} h^{\alpha\beta} \right] E_{\alpha\beta}$ is a spatial, symmetric and trace-free tensor.

In the case of the vacuum state we have $\rho = p = 0$, $T_{\mu\nu} \equiv 0$, and consequently $S_{\mu\nu} = 0$. Therefore the field equation describing a static brane takes the form

$$R_{\mu\nu} = -E_{\mu\nu} + \Lambda g_{\mu\nu}, \tag{10}$$

with the trace R of the Ricci tensor $R_{\mu\nu}$ satisfying the condition $R = R_{\mu}^{\mu} = 4\Lambda$.

In the vacuum case $E_{\mu\nu}$ satisfies the constraint $D_{\nu} E_{\mu}{}^{\nu} = 0$. In an inertial frame at any point on the brane we have $u^{\mu} = \delta_0^{\mu}$ and $h_{\mu\nu} = \mathrm{diag}(0, 1, 1, 1)$. In a static vacuum $Q_{\mu} = 0$ and the constraint for $E_{\mu\nu}$ takes the form [8]

$$\frac{1}{3} D_{\mu} U + \frac{4}{3} U A_{\mu} + D^{\nu} P_{\mu\nu} + A^{\nu} P_{\mu\nu} = 0, \tag{11}$$

where $A_{\mu} = u^{\nu} D_{\nu} u_{\mu}$ is the 4-acceleration. In the static spherically symmetric case we may chose $A_{\mu} = A(r) r_{\mu}$ and $P_{\mu\nu} = P(r) \left(r_{\mu} r_{\nu} - \frac{1}{3} h_{\mu\nu} \right)$, where $A(r)$ and $P(r)$ (the "dark

pressure" although the name dark anisotropic stress might be more appropriate) are some scalar functions of the radial distance r, and r_μ is a unit radial vector [13].

B. The motion of particles in stable circular orbits on the brane

In brane world models test particles are confined to the brane. Mathematically, this means that the equations governing the motion are the standard 4d geodesic equations [12, 14]. However, the bulk has an effect on the motion of the test particles on the brane via the metric. Since the projected Weyl tensor effectively serves as an additional matter source, the metric is affected by these bulk effects, and so are the geodesic equations. This is in contrast to Kaluza-Klein theories, where matter travels on 5d geodesics.

In order to obtain results which are relevant to the galactic dynamics, in the following we will restrict our study to the static and spherically symmetric metric given by

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\Omega^2, \tag{12}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

The Lagrangian $\mathcal{L}$ for a massive test particle traveling on the brane reads

$$\mathcal{L} = \frac{1}{2}\left(-e^{\nu}\dot{t}^2 + e^{\lambda}\dot{r}^2 + r^2\dot{\Omega}^2\right), \tag{13}$$

where the dot means differentiation with respect to the affine parameter.

Since the metric tensor coefficients do not explicitly depend on t and Ω, the Lagrangian (13) yields the following conserved quantities (generalized momenta):

$$-e^{\nu(r)}\dot{t} = E, \qquad r^2\dot{\Omega} = L, \tag{14}$$

where E is related to the total energy of the particle and L to the total angular momentum. With the use of conserved quantities we obtain from Eq. (13) the geodesic equation for massive particles in the form

$$e^{\nu+\lambda}\dot{r}^2 + e^{\nu}\left(1 + \frac{L^2}{r^2}\right) = E^2, \tag{15}$$

The second term of the left-hand side can, in some cases, be interpreted as an effective potential. For instance, for the Schwarzschild space-time, where $e^{\nu+\lambda} = 1$, the kinetic term is position independent. In that case the notion of an effective potential is appropriate. In

other cases, even one can still compute the turning points of the kinetic term, however, the effective potential interpretation is lost.

For the case of the motion of particles in circular and stable orbits the 'potential' must satisfy the following conditions: a) $\dot{r} = 0$ (circular motion) b) $\partial V_{eff}/\partial r = 0$ (extreme motion) and c) $\partial^2 V_{eff}/\partial r^{\ 2}\ |_{extr} > 0$ (stable orbit), respectively. Conditions a) and b) immediately give the conserved quantities as

$$E^2 = e^{\nu}\left(1 + \frac{L^2}{r^2}\right), \tag{16}$$

and

$$\frac{L^2}{r^2} = \frac{r\nu'}{2} e^{-\nu} E^2, \tag{17}$$

respectively. Equivalently, these two equations can be rewritten as

$$E^2 = \frac{e^{\nu}}{1 - r\nu'/2}, \qquad L^2 = \frac{r^3\nu'/2}{1 - r\nu'/2}. \tag{18}$$

We define the tangential velocity v_{tg} of a test particle on the brane, as measured in terms of the proper time, that is, by an observer located at the given point, as [26]

$$v_{tg}^2 = e^{-\nu} r^2 \left(\frac{d\Omega}{dt}\right)^2 = \frac{e^{\nu}}{E^2}\frac{L^2}{r^2}. \tag{19}$$

By using the constants of motion, and eliminating the quantity L, we obtain the expression of the tangential velocity of a test particle in a stable circular orbit [25] on the brane as

$$v_{tg}^2 = \frac{r\nu'}{2}. \tag{20}$$

Let us emphasize again that the function ν' is obtained by solving the field equations containing the bulk effects as additional matter terms; we consider this in Section III.

C. The gravitational field equations for a static spherically symmetric brane

With the metric given by (12) the gravitational field equations and the effective energy-momentum tensor conservation equation in the vacuum take the form [16, 17]

$$-e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} = \frac{48\pi G}{k^4\lambda_b} U + \Lambda, \tag{21}$$

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = \frac{16\pi G}{k^4\lambda_b}\left(U + 2P\right) - \Lambda, \tag{22}$$

$$e^{-\lambda}\frac{1}{2}\left(\nu''+\frac{\nu'^2}{2}+\frac{\nu'-\lambda'}{r}-\frac{\nu'\lambda'}{2}\right)=\frac{16\pi G}{k^4\lambda_b}\left(U-P\right)-\Lambda, \tag{23}$$

$$\nu'=-\frac{U'+2P'}{2U+P}-\frac{6P}{r\left(2U+P\right)}, \tag{24}$$

where $'=d/dr$. In the following we denote

$$\alpha=\frac{16\pi G}{k^4\lambda_b}. \tag{25}$$

As for the motion of the test particle on the brane we assume that they follow stable circular orbits, with tangential velocities given by Eq. (20). Thus, the rotational velocity of the test body is determined by the metric coefficient $\exp\left(\nu\right)$ only.

The field equations (21)–(22) yield the following effective energy density, radial and orthogonal pressure

$$\rho^{\text{eff}} = 3\alpha U+\Lambda, \tag{26}$$

$$P^{\text{eff}} = \alpha U+2\alpha P-\Lambda, \tag{27}$$

$$P_{\perp}^{\text{eff}} = \alpha U-\alpha P-\Lambda, \tag{28}$$

which by rewriting become

$$\rho^{\text{eff}}-P^{\text{eff}}-2P_{\perp}^{\text{eff}}=4\Lambda=\text{const}. \tag{29}$$

This is expected for the 'radiation' like source, the projection of the bulk Weyl tensor, which is trace-less $E_{\mu}^{\mu}=0$.

III. STRUCTURE EQUATIONS OF THE VACUUM IN THE BRANE WORLD MODELS

Eq. (21) can immediately be integrated to give

$$e^{-\lambda}=1-\frac{C_1}{r}-\frac{GM_U\left(r\right)}{r}-\frac{\Lambda}{3}r^2, \tag{30}$$

where C_1 is an arbitrary constant of integration, and we denoted

$$GM_U\left(r\right)=3\alpha\int_0^r U(r)r^2dr. \tag{31}$$

The function M_U is the gravitational mass corresponding to the dark radiation term (the dark mass). For $U=0$ the metric coefficient given by Eq. (30) must tend to the

standard general relativistic Schwarzschild metric coefficient, which gives $C_1 = 2GM$, where $M =$ constant is the baryonic (usual) mass of the gravitating system.

By substituting ν' given by Eq. (24) into Eq. (22) and with the use of Eq. (30) we obtain the following system of differential equations satisfied by the dark radiation term U, the dark pressure P and the dark mass M_U, describing the vacuum gravitational field, exterior to a massive body, in the brane world model [16]:

$$\frac{dU}{dr} = -\frac{(2U+P)\left[2GM + GM_U + \alpha\left(U+2P\right)r^3\right] - \frac{2}{3}\Lambda r^2}{r^2\left(1 - \frac{2GM}{r} - \frac{M_U}{r} - \frac{\Lambda}{3}r^2\right)} - 2\frac{dP}{dr} - \frac{6P}{r}, \tag{32}$$

$$\frac{dM_U}{dr} = \frac{3\alpha}{G}r^2U. \tag{33}$$

The tangential velocity can be written as

$$v_{tg}^2 = \frac{1}{2}\frac{2GM + GM_U + \alpha\left(U+2P\right)r^3 - \frac{2}{3}\Lambda r^2}{r\left(1 - \frac{2GM}{r} - \frac{GM_U}{r} - \frac{\Lambda}{3}r^2\right)}. \tag{34}$$

To close the system a supplementary functional relation between one of the unknowns U, P, M_U and v_{tg} is needed. Once this relation is known, Eqs. (32)–(20) give a full description of the geometrical properties and of the motion of the particles on the brane.

The system of equations (32) and (33) can be transformed to an autonomous system of differential equations by means of the transformations

$$q = \frac{2GM}{r} + \frac{GM_U}{r} + \frac{\Lambda}{3}r^2, \qquad \mu = 3\alpha r^2U + 3r^2\Lambda,$$

$$p = 3\alpha r^2P - 3r^2\Lambda, \qquad \theta = \ln r. \tag{35}$$

We shall call μ and p the "reduced" dark radiation and pressure, respectively.

With the use of the new variables given by Eqs. (35), Eqs. (32) and (33) become

$$\frac{dq}{d\theta} = \mu - q, \tag{36}$$

$$\frac{d\mu}{d\theta} = -\frac{(2\mu+p)\left[q + \frac{1}{3}(\mu+2p)\right]}{1-q} - 2\frac{dp}{d\theta} + 2\mu - 2p. \tag{37}$$

Eqs. (32) and (33), or, equivalently, (36) and (37) may be called the structure equations of the vacuum on the brane. In order to close this system an "equation of state", relating the reduced dark radiation and the dark pressure terms is needed. Generally, this equation of state is given in the form $P = P(U)$.

In the new variables the tangential velocity of a particle in a stable circular orbit on the brane is given by

$$v_{tg}^2 = \frac{1}{2}\frac{q+\frac{1}{3}(\mu+2p)}{1-q}. \tag{38}$$

By using the expression of the tangential velocity, Eq. (37) can be rewritten as

$$\frac{d}{d\theta}(\mu+2p) = -2(2\mu+p)v_{tg}^2 + 2\mu - 2p. \tag{39}$$

Eq. (39) allows the easy check of the physical consistency of some simple equations of state for the dark pressure. The equation of state $\mu + 2p = 0$ immediately gives $v_{tg}^2 = 1$, implying that all test particle in stable circular motion on the brane move at the speed of light, a fact that is contradicted by the observations at the galactic scale. The equation of state $2\mu + p = 0$ gives $\mu = \mu_0/r^2$, where $\mu_0 =$ constant is an arbitrary integration constant, $U = \mu_0/3\alpha r^4$ and $GM_U = -\mu_0/3r^3$, respectively. In the limit of large r, the tangential velocity v_{tg}^2 tends to zero, $v_{tg} \to 0$. Therefore, this model seems also to be ruled out by observations. The case $\mu = p$ gives $\mu(\theta) = \mu_0 \exp\left[-2\int v_{tg}^2(\theta)d\theta\right]$, $\mu_0 =$ constant, and $q(\theta) = \left(2v_{tg}^2 - \mu\right)/\left(1+2v_{tg}^2\right)$.

The dark radiation and the dark pressure can be obtained as a functions of the tangential velocity in a closed analytical form in the important case of a linear equation of state of the form

$$p = (\Gamma - 2)\mu + \beta, \tag{40}$$

with Γ and β arbitrary constants. Then the reduced dark radiation can be obtained as

$$\begin{aligned}\mu(\theta) = {} & \theta^{2(3-\Gamma)/(2\Gamma-3)} \exp\left[-\frac{2\Gamma}{2\Gamma-3}\int v_{tg}^2(\theta)\,d\theta\right] \\ & \times \left\{C_0 - \frac{3\beta}{2\Gamma-3}\int\left[1+v_{tg}^2(\theta)\right]\theta^{-2(3-\Gamma)/(2\Gamma-3)}\exp\left[\frac{2\Gamma}{2\Gamma-3}\int v_{tg}^2(\theta)\,d\theta\right]\right\},\end{aligned} \tag{41}$$

where C_0 is an arbitrary integration constant. Hence, if the velocity profile of a test particle in stable circular motion is known, one can obtain all the relevant physical parameters for a static spherically symmetric system on the brane.

IV. DARK RADIATION, DARK PRESSURE AND THE GALACTIC METRIC IN THE CONSTANT TANGENTIAL VELOCITY REGION

The galactic rotation curves provide the most direct method of analyzing the gravitational field inside a spiral galaxy. The rotation curves have been determined for a great number of

spiral galaxies. They are obtained by measuring the frequency shifts z of the light emitted from stars and from the 21cm radiation emission from the neutral gas clouds. Usually, the astronomers report the resulting z in terms of a velocity field v_{tg}. The observations show that at distances large enough from the galactic center

$$v_{tg} \approx 200 - 300\text{km/s} = \text{constant}. \tag{42}$$

This behavior has been observed for a large number of galaxies [1].

In the following we use this observational constraint to reconstruct the metric of a galaxy on the brane. The constancy of v_{tg} allows us to express the function q from Eq. (20) as

$$q = \frac{2v_{tg}^2}{1+2v_{tg}^2} - \frac{1}{3\left(1+2v_{tg}^2\right)}\left(\mu + 2p\right), \tag{43}$$

giving immediately

$$\frac{dq}{d\theta} = -\frac{1}{3\left(1+2v_{tg}^2\right)}\left(\frac{d\mu}{d\theta} + 2\frac{dp}{d\theta}\right). \tag{44}$$

From Eq. (39) we obtain

$$\frac{d\mu}{d\theta} + 2\frac{dp}{d\theta} = -2\left(1-2v_{tg}^2\right)\mu - 2\left(1+v_{tg}^2\right)p, \tag{45}$$

which can be used to simplify Eq. (44) into

$$\frac{dq}{d\theta} = -\frac{2}{3\left(1+2v_{tg}^2\right)}\left[\left(1-2v_{tg}^2\right)\mu - \left(1+v_{tg}^2\right)p\right]. \tag{46}$$

By eliminating $dq/d\theta$ between Eqs. (36), and (37) and by using the expression of q given by Eq. (43), we obtain the equation of state for the dark pressure in the constant tangential velocity region as

$$p = \frac{3+v_{tg}^2}{v_{tg}^2}\mu - 3. \tag{47}$$

This shows that, as a function of the "reduced" dark radiation, in the coordinate θ, the "reduced" dark pressure obeys a generalized linear equation of state.

By using the generalized linear equation of state, Eq. (37) can be written as

$$\frac{d\mu}{d\theta} = -n\mu + m, \tag{48}$$

where

$$n = \frac{2v_{tg}^2}{3\left(2+v_{tg}^2\right)}\left[\frac{3+v_{tg}^2}{v_{tg}^2} + 3v_{tg}^2 + 2\right], \tag{49}$$

and

$$m=\frac{2\left(1+v_{tg}^{2}\right)v_{tg}^{2}}{2+v_{tg}^{2}}. \tag{50}$$

Eq. (48) can be immediately integrated to give

$$\mu\left(\theta\right)=Ce^{-n\theta}+\frac{m}{n}, \tag{51}$$

with C an arbitrary constant of integration.

In the standard radial coordinate r the "reduced" dark radiation energy density is given by

$$\mu(r)=\frac{C}{r^{n}}+\frac{\left(1+v_{tg}^{2}\right)v_{tg}^{2}}{1+v_{tg}^{2}+v_{tg}^{4}}, \tag{52}$$

while the "reduced" dark pressure is

$$p(r)=\frac{3+v_{tg}^{2}}{v_{tg}^{2}}\frac{C}{r^{n}}-\frac{2v_{tg}^{2}}{1+v_{tg}^{2}}. \tag{53}$$

From these equations we find the variation of the dark radiation and dark pressure in the constant tangential velocity region as

$$U(r)=\frac{1}{3\alpha}\left[\frac{C}{r^{n+2}}+\frac{\left(1+v_{tg}^{2}\right)v_{tg}^{2}}{1+v_{tg}^{2}+v_{tg}^{4}}\frac{1}{r^{2}}-3\Lambda\right], \tag{54}$$

and

$$P(r)=\frac{1}{3\alpha}\left[\frac{3+v_{tg}^{2}}{v_{tg}^{2}}\frac{C}{r^{n+2}}-\frac{2v_{tg}^{2}}{1+v_{tg}^{2}}\frac{1}{r^{2}}+3\Lambda\right], \tag{55}$$

respectively.

The metric tensor component $\exp(\nu)$ can be found from the expression of the tangential velocity as

$$e^{\nu}=C_{2}r^{2v_{tg}^{2}}, \tag{56}$$

with C_2 an arbitrary constant of integration, while $\exp(\lambda)$ is given by

$$e^{-\lambda}=1-\frac{2GM}{r}-\frac{C}{\left(1-n\right)r^{n}}-\frac{\left(1+v_{tg}^{2}\right)v_{tg}^{2}}{1+v_{tg}^{2}+v_{tg}^{4}}-\frac{\Lambda}{3}r^{2}. \tag{57}$$

The function $M_U(r)$, which gives the mass associated with the dark radiation, is given by

$$GM_{U}(r)=\frac{C}{\left(1-n\right)r^{n-1}}+\frac{\left(1+v_{tg}^{2}\right)v_{tg}^{2}}{1+v_{tg}^{2}+v_{tg}^{4}}r. \tag{58}$$

Hence, the problem of the determination of the metric and of the dark radiation and dark pressure in the constant tangential velocity region around galaxies in the brane world model

is completely solved. It remains to determine the values of the integrations constants C and C_2.

To do this we assume that in the presence of matter the bulk effects on the brane are negligible small. This definitely represents a very good approximation for the galaxy, which has as only constituent "normal" matter, with a density much higher than the density of the matter outside the galaxy. On the other hand, at large distances from the galactic center the cosmological background dominates the dynamics.

Therefore, by assuming that the radius of the baryonic matter distribution in the galaxy is R, we can match our solution for $r = R$ with the Schwarzschild-de Sitter metric, so that

$$e^{\nu}|_{r=R} \approx 1 - \frac{2GM}{R} - \frac{\Lambda}{3}R^2, \tag{59}$$

and

$$e^{\lambda}|_{r=R} \approx \left(1 - \frac{2GM}{R} - \frac{\Lambda}{3}R^2\right)^{-1}, \tag{60}$$

respectively. This gives

$$C_2 = R^{-2v_{tg}^2}\left(1 - \frac{2GM}{R}\right), \tag{61}$$

and

$$C = -(1-n)\frac{\left(1+v_{tg}^2\right)v_{tg}^2}{1+v_{tg}^2+v_{tg}^4}R^n, \tag{62}$$

respectively. Therefore the dark mass, the dark radiation and the dark pressure terms can finally be written as

$$GM_U(r) = \frac{\left(1+v_{tg}^2\right)v_{tg}^2}{1+v_{tg}^2+v_{tg}^4}r\left[1-\left(\frac{R}{r}\right)^n\right], \tag{63}$$

$$\alpha U(r) = \frac{1}{r^2}\left\{\frac{\left(1+v_{tg}^2\right)v_{tg}^2}{3\left(1+v_{tg}^2+v_{tg}^4\right)}\left[1-(1-n)\left(\frac{R}{r}\right)^n\right] - \Lambda r^2\right\}, \tag{64}$$

$$\alpha P(r) = -\frac{1}{r^2}\left[\frac{(1-n)\left(1+v_{tg}^2\right)\left(3+v_{tg}^2\right)}{3\left(1+v_{tg}^2+v_{tg}^4\right)}\left(\frac{R}{r}\right)^n - \Lambda r^2\right]. \tag{65}$$

Hence, the components of the Weyl tensor from the bulk can be obtained in terms of the observable quantities, like the baryonic radius of the galaxy and the tangential velocity of test particles gravitating in stable circular orbits around the galactic center.

The brane world metric in the constant tangential velocity region has a singularity at some $r = r_h$, which can be obtained by solving the non-linear algebraic equation $\exp(-\lambda) = 0$. Depending on the numerical values of n this equation may have one or more positive roots, which define an event horizon. Therefore the singularity at $r = 0$ is hidden and cannot be seen by an external observer.

V. QUALITATIVE ANALYSIS OF THE STRUCTURE EQUATIONS OF THE VACUUM ON THE BRANE FOR GIVEN SIMPLE EQUATIONS OF STATE OF THE DARK PRESSURE

In the present Section we investigate the general behavior of the tangential velocity on the brane by assuming a simple linear equation of state for the dark pressure. Since generally the structure equations of the vacuum on the brane cannot be solved exactly, we shall analyze the dynamics of test particles by using methods from the qualitative analysis of dynamical systems, by closely following the approach of [27].

We consider the case in which the dark pressure is proportional to the dark radiation $P = \gamma U$, where γ is an arbitrary constant, which can take both positive and negative values. In the reduced variables μ and p the linear equation of state is $p = \gamma\mu$, and the structure equations of the gravitational field on the brane have the form

$$\frac{dq}{d\theta} = \mu - q, \tag{66}$$

$$(1+2\gamma)\frac{d\mu}{d\theta} = 2(1-\gamma)\mu - \frac{(\gamma+2)\mu\left[q + \frac{1+2\gamma}{3}\mu\right]}{1-q}. \tag{67}$$

The tangential velocity of the particles in stable circular orbit is given by

$$v_{tg}^2 = \frac{1}{2}\frac{q + \frac{1+2\gamma}{3}\mu}{1-q}. \tag{68}$$

Let us firstly analyze the special case where $\gamma = -1/2$. Then, by virtue if the second Eq. (67), we obtain two possible solution, either $q = \mu = 0$ or $q = \mu = 2/3$. Since both solution are physically uninteresting, implying either $v_{tg} = 0$ or $v_{tg} = 1$, let us assume henceforth that $\gamma \neq -1/2$, and rewrite the system of equation into the following form

$$\frac{dq}{d\theta} = -q + \mu, \tag{69}$$

$$\frac{d\mu}{d\theta} = \frac{2(1-\gamma)}{1+2\gamma}\mu - \frac{\gamma+2}{1+2\gamma}\frac{\mu\left[q + \frac{1+2\gamma}{3}\mu\right]}{1-q}, \tag{70}$$

which is finally written as

$$\frac{d\xi}{d\theta} = A\xi + n, \tag{71}$$

where we have denoted

$$\xi = \begin{pmatrix} q \\ \mu \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 1 \\ 0 & 2(1-\gamma)/(1+2\gamma) \end{pmatrix}, \quad n = \begin{pmatrix} 0 \\ \frac{\gamma+2}{1+2\gamma}\frac{\mu\left[q+\frac{1+2\gamma}{3}\mu\right]}{1-q} \end{pmatrix} \tag{72}$$

The system of equations (71) has two critical points $X_0 = (0,0)$ and $X_\gamma = (3(1-\gamma)/(\gamma^2+\gamma+7), 3(1-\gamma)/(\gamma^2+\gamma+7))$. For $\gamma = 1$, the two critical points of the system coincide. Depending of the values of γ, these points lie in different regions of the phase space plane (q,μ).

Since the term $||n||/||\xi|| \to 0$ as $||\xi|| \to 0$, the system of equations (71) can be linearized at the critical point X_0. The two eigenvalues of the matrix A are given by $r_1 = -1$ and $r_2 = 2(1-\gamma)/(1+2\gamma)$, and determine the characteristics of the critical point X_0. For $\gamma \in (-\infty, -1/2) \cup [1, \infty)$ both eigenvalues are negative and unequal. Therefore, for such values of γ the point X_0 is an improper asymptotically stable node.

If $\gamma \in (-1/2, 1)$, we find one positive and one negative eigenvalue, which corresponds to an unstable saddle point at the point X_0. Moreover, since the matrix $dA/d\xi(X_0)$ has real non-vanishing eigenvalues, the point X_0 is hyperbolic. This implies that the properties of the linearized system are also valid for the full non-linear system near the point X_0. It should be mentioned however, that this first critical point is the less interesting one from a physical point of view, since it corresponds to the 'trivial' case where both physical variables vanish.

Before performing a similar analysis for the second critical point, it is worth mentioning that the structure equations can be solved exactly for the value $\gamma = -2$. In that case, the non-linear term n in Eq. (71) identically vanishes, and the system of equations becomes a simple linear system of differential equations. For $\gamma = -2$ the two eigenvalues of A are given by $r_1 = -1$ and $r_2 = -2$, the two corresponding eigenvectors are linearly independent and the general solution can be written as follows

$$\xi_{\gamma=-2} = (q_0 + \mu_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\theta} + \mu_0 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2\theta}, \tag{73}$$

where $q_0 = q(0)$ and $\mu_0 = \mu(0)$. One can easily transform this solution back into the radial coordinate r form by using $\theta = \log(r)$ and get

$$\mu_{\gamma=-2} = \frac{\mu_0}{r^2}, \quad q_{\gamma=-2} = \frac{q_0}{r} + \mu_0 \left(\frac{1}{r} - \frac{1}{r^2}\right). \tag{74}$$

This solution can be used to analyze the resulting tangential velocity profile as a function of the radius r, which turn out to have the same form as predicted by the Newtonian analysis. Therefore such an equation of state, namely $p = -2\mu$, cannot account for flat rotation curves in galaxies, see the end of Section III.

Let us now analyze the qualitative behavior of the second critical point X_γ. To do this, one has to Taylor expand the right-hand sides of Eqs. (69) and (70) around X_γ and obtain the matrix $\tilde{A}$ which corresponds to the system, linearized around X_γ. This linearization is again allowed since the resulting non-linear term, $\tilde{n}$ say, also satisfies the condition $||\tilde{n}||/||\xi|| \to 0$ as $||\xi|| \to X_\gamma$. The resulting matrix reads

$$\tilde{A} = \begin{pmatrix} -1 & 1 \\ \frac{3(-\gamma^2+5\gamma-4)}{(2+\gamma)^2(1+2\gamma)} & \frac{\gamma-1}{2+\gamma} \end{pmatrix} \tag{75}$$

and its two eigenvalues are given by

$$r_\pm = \frac{-3-6\gamma \pm \sqrt{16\gamma^4+8\gamma^3+132\gamma^2-28\gamma-47}}{4\gamma^2+10\gamma+4}. \tag{76}$$

For $-0.5 < \gamma < 0.674865$ the argument of the square root becomes negative and the eigenvalues complex. Moreover, the values $\gamma = -1/2$ and $\gamma = -2$ have to be excluded, since the eigenvalues in Eq. (75) are not defined in these cases. However, both cases have been treated separately above.

If $\gamma \in (-\infty, -1/2) \cup (1, \infty)$, then X_γ corresponds to an unstable saddle point and for $\gamma \in (0.674865, 1)$ it corresponds to an asymptotically stable improper node. More interesting is the parameter range $\gamma \in (-1/2, 0.674865)$ where the eigenvalues become complex, however, their real parts are negative definite. Hence, for those values we find an asymptotically stable spiral point at X_γ. Since this point is also an hyperbolic point, the described properties are also valid for the non-linear system near that point, see Fig. 1. This qualitative study of the behavior of the structure equations of the vacuum on the brane shows that not all equations of state of the dark pressure would predict flat rotation curves.

With Eq. (47) we were able to explicitly construct the equation of state that yields rotation curves of constant velocity. Since these velocities are about $200 - 300$km/s, we obtain as an equation of state

$$p \approx (3-7) \times 10^6 \mu - 3, \tag{77}$$

which is of the general form $p = a\mu - b$.

The above qualitative analysis can easily be repeated for the physically relevant equation of state. The first difference is that $X_0 = (0,0)$ is no longer a critical point to the system, and the critical point X_γ gets a contribution due to the parameter b. Moreover, X_γ is replaced

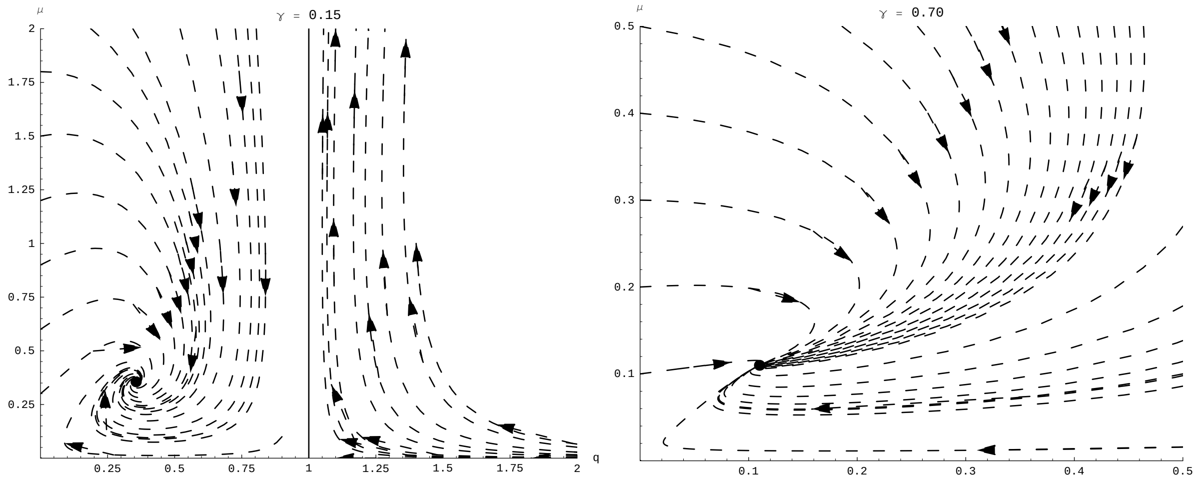

FIG. 1: The left figure shows the phase space plot $\mu = \mu(q)$ of the system (69)–(70) for $\gamma = 0.15$. The dot represents the critical point X_γ and one clearly sees that X_γ corresponds to an asymptotically stable spiral point. In the right figure $\gamma = 0.70$, and there the critical point X_γ corresponds to an asymptotically stable improper node.

by a pair of critical points, related to the two square roots of a quadratic equation that reduces to a linear equations as $a \to 0$. The critical point in the $p > 0$ and $\mu > 0$ region in phase space corresponds to an asymptotically stable improper node, and the qualitative picture is that of the right plot of Fig. 1.

VI. DISCUSSIONS AND FINAL REMARKS

The galactic rotation curves and the mass distributions in clusters of galaxies continue to pose a challenge to present day physics. One would like to have a better understanding of some of the intriguing phenomena associated with them, like their universality and the very good correlation between the amount of dark matter and the luminous matter in the galaxy. To explain these observations, the most commonly considered models are based on particle physics in the framework of Newtonian gravity, or of some extensions of general relativity [6].

In the present paper we have considered, and further developed, an alternative view to the dark matter problem [17, 18], namely, the possibility that the galactic rotation curves and the mass discrepancy in clusters of galaxies can naturally be explained in models in which our Universe is a brane in a multi-dimensional spacetime. The extra terms in the

gravitational field equations on the brane induce a supplementary gravitational interaction, which can account for the observed behavior of the galactic rotation curves. As one can see from Eq. (63), in the limit of large $r \gg R$, and by taking into account that $v_{tg}^2 \ll 1$, we obtain $GM_U \approx v_{tg}^2 r$, a behavior which is perfectly consistent with the observational data [1], and is usually attributed to the existence of the dark matter. Since inside the galaxy $M_U \approx 0$, it follows that the constant tangential velocity of the test particles is determined by the baryonic mass M_B and the radius R of the galaxy by the equation $v_{tg}^2 \approx GM_B/R$. This gives for the dark mass the scaling relation

$$\frac{M_U}{M_B} \approx \frac{r}{R}. \tag{78}$$

By using the simple observational fact of the constancy of the galactic rotation curves, the galactic metric and the corresponding components of the projected Weyl tensor $E_{\mu\nu}$ (dark radiation and dark pressure) can be completely reconstructed, without any supplementary assumption.

Weak lensing provides a unique probe of the gravitational potential on large scales. Hence, in the context of dark matter, it can provide constrains on the extent and shapes of dark matter halos. Furthermore, it can test alternative theories of gravity (without dark matter). Due to the fixed form of the galactic metric on the brane, in our model the light bending angle is a function of the tangential velocity of particles in stable circular orbit and of the baryonic mass and radius of the galaxy. The specific form of the bending angle is determined by the brane galactic metric, and this form of the metric is different as compared to the other dark matter models (long-range self-interacting scalar fields, MOND, non-symmetric gravity etc.). As shown in [18, 19, 23, 24], the gravitational light deflection angle is much larger than the value predicted by the standard general relativistic approach. Even when we compare our results with standard dark matter models, like the isothermal dark matter halo model, we still may find significant differences in the lensing effect. Therefore the study of the gravitational lensing may provide the evidence for the existence (or non-existence) of the bulk effects on the brane. On the other hand, since in this model there is only baryonic matter, all the physical properties at the galactic level are determined by the amount of the luminous matter and its distribution.

To test alternative theories of gravity by using lensing one can use two approaches [28]. A measurement of the radial dependence of the lensing signal gives the best accuracy. This

method requires the knowledge of the deflection law. The most direct test is the detection of the azimuthal variation of the lensing signal around the lenses, and the measurements does not require knowledge of the deflection law. Any anisotropy in the lensing signal caused by the galaxy decreases as r^{-2} and therefore is negligible in the bulk effects dominated region.

The detection of an anisotropy in the lensing signal around galaxies may provide the best test for discriminating between dark matter and alternative gravity models, including the present model. The results of a study of weak lensing by galaxies based on 45.5 deg^2 of R_C band imaging data from the Red-Sequence Cluster Survey were presented in [29]. A significant flattening was detected, which implies that the halos are well aligned with the light distribution, and an isotropic lensing signal is excluded with 99.5% confidence.

From an observational point of view an important problem is to estimate an upper bound for the cutoff of the constancy of the tangential velocities. The idea is to consider the point at which the decaying density profile of the dark radiation associated to the galaxy becomes smaller than the average energy density of the Universe, and the cosmological expansion dominates the dynamics of the particles. Let the value of the coordinate radius at the point where the two densities are equal be R_U. Then at this point $3\alpha\, U(R_U) = (8\pi G)\rho_{univ}$, where ρ_{univ} is the mean energy density of the universe.

An alternative estimation of R_U can be obtained from the observational requirement that at the cosmological level the energy density of the dark matter (which in our case is the dark mass associated to the dark radiation) represents a fraction $\Omega_{DM} \approx 0.25$ of the total energy density of the flat universe, with density parameter $\Omega = 1$. Therefore the "dark matter" contribution inside a ball of radius R_U is given by $4\pi\Omega_{DM} R_U^3 \rho_{crit}/3$, which gives

$$R_U \approx \frac{1}{\sqrt{2\Omega_{DM}}} \frac{1}{h(z) H_0} v_{tg}, \tag{79}$$

where $h(z)$ is the Hubble constant normalized to its local value: $h^2(z) = \Omega_m (1+z)^3 + \Omega_\Lambda$, Ω_m is the total mass density parameter and Ω_Λ is the dark energy density parameter. For a tangential velocity of the order of 200 km/s it follows that $R_U \approx 2.8$ Mpc. Therefore, we predict that the flat rotation curves region should extend up to distances of a few megaparsecs from the galactic center.

If we assume that the flat rotation curves extend indefinitely, the resulting spacetime is not asymptotically flat, but of de Sitter type. This is due to the presence of the cosmological constant Λ on the brane. Observationally, the galactic rotation curves remain flat to the

farthest distances that can be observed. Therefore, from both the observational and the theoretical point of view it is important to estimate the possible role of the cosmological constant on the extra galactic dynamic. By assuming that the cosmological constant has a numerical value of the order of $\Lambda \approx 3 \times 10^{-56}$ cm^{-2} [30], at a distance of $r = 100$ kpc, for a galaxy with mass $M = 10^{10} M_{\odot}$, the quantities $GM/r \approx 4.94 \times 10^{-9}$ and $\Lambda r^2 \approx 2.7 \times 10^{-9}$ are roughly of the same order of magnitude. For clusters of galaxies with masses of the order of $10^{14} M_{\odot}$ and radii of the order of 2 Mpc, we have $GM/r \approx 2.47 \times 10^{-6}$ and $\Lambda r^2 \approx 1.08 \times 10^{-6}$, respectively.

Hence the presence of a non-zero cosmological constant may play an important role in the dynamics of the test particles in stable circular orbits at the outer boundary of the galaxies or clusters of galaxies, since it may generate physical effects that are of the same order of magnitude as the extra-dimensional effects from the bulk. Therefore, for the correct estimation in regions far away from the galactic center of the dark radiation, of the dark pressure, of the tangential velocities of test particles and of other related effects (like, for example, the bending angle of the light) one must take into account the effect of the cosmological constant on the brane.

In the present model all the relevant physical quantities, including the dark mass (playing the role of the dark matter on the brane), the dark radiation and the dark pressure describing the non-local effects due to the gravitational field of the bulk, are expressed in terms of observable parameters – the tangential velocity, the baryonic mass and the radius of the galaxy.

There are three observationally testable predictions of the model: a) the constant rotation curve region may extend much farther than presently observed, up to a few megaparsecs, b) the weak lensing effect in the dark radiation dominated region is different from that predicted by the standard dark matter models, and other alternative gravitational theories and c) the numerical value of the maximum rotational velocity is mainly determined by the "normal" matter content of the galaxy.

As for the main advantages of our model, on the one hand it is esthetically attractive – "matter" is a manifestation of geometry, it is relatively simple and, on the other hand, it is directly testable observationally. The best testing ground for brane world models is the extra-galactic space, where the effects of galactic/extra-galactic matter perturbations are minimal, and where we expect that the bulk effects appear in a "pure" form, and they

dominate the dynamics of test particles (gas clouds). Moreover, the extra-galactic space is accessible to direct optical or radio observation. In the solar system the bulk effects are very small, and difficult to observe.

Therefore, these results open the possibility of testing the brane world models by using direct astronomical and astrophysical observations at the galactic or extra galactic scale. In this paper we have provided some basic theoretical tools necessary for the in depth comparison of the predictions of brane world models and of the observational results.

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